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Survey and Comparison of Engineering Beam Theories for Helicopter Rotor Blades

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Nomenclature

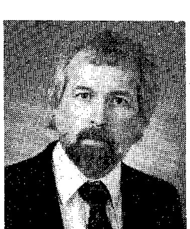
A = blade cross-sectional area
B_1, B_2 = blade cross-sectional constants
E = Young's modulus
e = blade root offset
e_A = tension axis chordwise offset
F = internal forces
f = applied forces
G = shear modulus
H = angular momentum
$I_{y'}$ = area moment of inertia about the deformed y axis
$I_{z'}$ = area moment of inertia about the deformed z axis
J = polar moment of inertia
K = moment strain
k_m = polar radius of gyration
k_{m1} = blade chordwise radius of gyration
k_{m2} = blade flapwise radius of gyration
L_{v} = in-plane load
L_{w} = out-of-plane load
M = internal moment
M_{φ} = torsion load
m = blade running mass, applied moment, Eq. (16)
P = linear momentum
q = generalized displacement
R = blade length
r = blade length coordinate
T = tension
u = blade extension
V = generalized linear speeds
v = in-plane blade bending
w = out-of-plane blade bending
γ = force strain
θ = blade twist
φ = blade torsion
ψ = generalized rotation
Ω = rotor speed, generalized angular speeds, Eq. (16)

Introduction

THE two purposes of this article are to review the development of engineering beam theories for helicopter rotor blades in its historic context, and to explain the characteristics and differences among the various formulations of this problem. By reviewing the discipline together with a view of history, it will be seen that each stage of development was mostly in reaction to new and more stringent requirements on theoretical analyses. Perhaps this tracing of the path from methods of the past to present solutions will provide a guide for anticipating future requirements. In addition, a review of the developments that have led to the current state-of-the-art will provide to the practicing engineer the scope of analytical methods from which he can choose the most appropriate for his application.

From the perspective of history, the development of engineering beam theories for helicopter rotor blades can be seen to parallel (or follow) the evolution of hub and blade design. The earliest helicopters used fully articulated or teetering hub (for two-bladed rotors) designs to relieve the blade root stresses, and were fitted with very stiff blades. As a result of having nearly all of the blade flexibility conentrated at the blade root, elastic bending was not considered to be a significant factor in the computation of blade response, loads, and rotor stability. With the introduction of hingeless designs, blade bending assumed a more important role in response and stability calculations. For a hingeless blade, the coupling between blade motions, which in articulated designs were almost entirely dictated by the blade root geometry, is dependent on the deformation geometry of the blade itself. Today, with the emphasis on bearingless rotors, accurate modeling of the elastic bending of the flexbeam is of paramount importance in blade calculations, since the flexbeam replaces the blade pitch bearing and transmits pitching motion through torsion of the flexbeam. In addition, there has been a move towards using composite materials to fabricate blades. Composite blades offer the advantages of longer fatigue lives and a greater ability to tailor their properties. However, the structural complexity that results from the use of anisotropic materials and nonsymmetric lay-ups introduces a need for increased complexity in the structural model. The coupling between blade deformations of composite blades is dependent on the elastic couplings built into the blade structure, as well as on the deformation of the blade.

The development of engineering beam theories for rotor blades has also been influenced by the revolutionary advances



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in digital computers. Forty years ago, when the helicopter industry was just coming out of its infancy, the vast majority of computations were performed using hand calculations and slide rules. As a result, every effort was made to keep the equations simple. The solution of nonlinear equations was too time-consuming to be of any practical value, and even systems of linear, algebraic equations had to be limited to a relatively small number of unknowns. Because of these computational limitations, modal methods that used a small number of shape functions to represent the deformed shape of the blade were the most practical and popular. These functions were usually obtained from the closed-form solutions of simpler systems of equations, such as those for nonrotating, pinned-free, or cantilever beams. With the introduction of digital computers into the engineering mainstream, it became practical to calculate the solution of nonlinear equations or large systems of linear equations. Thus, it became practical to include nonlinearities in the blade equations of motion. In addition, these capabilities fueled the popularity of finite-element methods, which improved the engineer's ability to model complex structures. The power and speed of computers have multiplied to the point where today, systems of nonlinear equations with hundreds of unknowns can be solved routinely. These advances in computational capabilities have opened up a myriad of possibilities for modeling structures and solving their equations of motion.

Evolution of Rotor Blade Analysis

In the early days of helicopter development, after the first flight of Igor Sikorsky's VS-300 in 1940, rotor blades were generally constructed using a tubular steel spar, plywood ribs, and either fabric or plywood covering. Analyses of rotor blade response and aeroelastic stability during this era assumed that these blades were essentially rigid.[1,2] In 1944, the first metal blades flew on the Hiller XH-44, but it was not until 1952 that the first production metal blades were delivered on the Sikorsky S-52. It had become apparent by this time that the assumption of rigid blades was not sufficient for the accurate calculation of blade response. One of the first analyses to include the effect of bending flexibility on blade response was formulated by Flax.[3] His analysis incorporated the effects of elastic, out-of-plane bending moments in the calculation of the blade flapping loads. He did not, however, include any contributions from blade bending in his calculations of the inertial and aerodynamic loads. At nearly the same time, Johnson and Mayne[4] concluded that out-of-plane blade bending had a significant effect on the blade inertial loads as well as producing elastic bending moments. The effect of bending deformations on the aerodynamic loads was again neglected. Following up on these findings, Flax and Goland[5] added the influence of blade bending deformations on the aerodynamic loads to the inertial and elastic loads, and produced a complete analysis for the out-of-plane response of a rotor blade.

While blade flapping is arguably the most predominant motion that a rotating blade undergoes, in-plane motions can also be important. The work discussed above considers only out-of-plane blade response, and neglects in-plane motion. One of the first to look at the effects of in-plane bending was Yuan,[6] who considered lag bending without flap bending. The groundwork for including both flap and lag motion in the analysis of rotating blades was laid by such authors as DiPrima and Handelman[7] and Shulman.[8] DiPrima and Handelman derived the equations of a rotating, elastic blade, but did not consider aerodynamic forces. Since Shulman was concerned with the flap-lag stability of rotors in high-speed forward flight, the influence of aerodynamics was more significant to his work.

The addition of the effect of torsion in the blade equations was first accomplished by Houbolt and Brooks.[9] To this point, blade bending and torsion had been considered, but only separately from one another. Their formulation of the equations of motion has been widely accepted and is frequently used as the basis for linear rotor blade analyses.

Driven by the development of hingeless hub designs, improvements on the Houbolt and Brooks formulation were made by Hodges and Dowell.[10] While the linear terms in their equations for blade equilibrium are nearly identical to those of Houbolt and Brooks, Hodges and Dowell have, through an ordering scheme, retained the dominant nonlinear terms in the equations of motion. This formulation of the rotor blade equations has also received widespread acceptance. Contributions to the development of the nonlinear equations of motion of rotor blades have also been made by Kaza and Kvaternik,[11] and Rosen and Friedmann.[12]

In the past few years, there has been a movement away from ordering schemes as a means to simplify the nonlinear beam equations of motion. Crespo da Silva[13] has shown that third-order terms can have a significant impact on aeroelastic stability calculations. However, using such an ordering scheme greatly increases the complexity of the resulting equations. By using a compact, matrix notation, Hodges[14] derived an implicit set of equations for a beam undergoing large rotations. Simo[15] (and later Hodges[16]) derived a set of explicit nonlinear beam equations using a mixed formulation. These equations allow large rotations to be treated exactly in the kinematic and equilibrium equations, while still expressing them in a compact form.

In the early sixties, experimental blades that used composite materials (as well as wood and metal) were designed. The seventies saw the development of the first production all-composite blade installed on a BO-105 helicopter by Messerschmidt-Boelkow-Blohm. With the increasing use of composite materials for rotor blades and bearingless rotor flexbeams, analyses had to be formulated to model the effects of anisotropic material properties. Analysis methods for composites were developed by Bauchau[17] and many others. These methods were then included in the analyses of rotating beams by investigators, including Bauchau and Hong[18] and Kosmatka and Friedmann.[19] Hodges[20] has put together an excellent overview of a wide variety of modeling approaches that have been used to analyze composite rotor blades.

Rotor Blade Analyses

In the following sections, a variety of rotor blade analyses will be described and compared. For the purposes of this article, these analyses have been separated into four groups: 1) linear analyses, 2) nonlinear analyses, 3) exact analyses, and 4) composite analyses. In this context, the terms linear, nonlinear, and exact refer primarily to the kinematics of blade motion, and do not include material properties or aerodynamic forces.

Linear Analyses

The analyses included in this section are those that either limit the scope of the analysis such that the equations of motion are linear, or include in the analysis only linear contributions to the nonlinear equations of motion. The progression from a single-degree-of-freedom flapping blade, to two-degree-of-freedom bending, to coupled bending and torsion is indicative of the advances that were made in linear analyses.

Linear equations of motion for blade flapping motion, such as those derived by Flax and Goland,[5] are sufficient in many engineering applications. Considering only the structural and inertial terms, this equation can be written as

$$-(Tw')' + m\ddot{w} + (EIw'')'' = 0 \tag{1}$$

where

$$T = \Omega^2 \int_r^R mx \, dx \tag{2}$$

Obviously, the only additions to the well-known Euler-Bernoulli equations for a nonrotating beam are the centrifugal stiffening terms.

DiPrima and Handleman[7] derived a set of linear equations of motion for a twisted, rotating beam undergoing bending deformations. In that derivation, the blades were allowed to have large twist angles, and the distribution of twist was permitted to be nonlinear. The existence of the Coriolis coupling terms between flap and lag was acknowledged, but neglected in order to arrive at a linear set of equations.

The best-known and most useful linear equations for a helicopter rotor were derived by Houbolt and Brooks.[9] Building on the previous formulation and adding the torsional elastic motion, this set of equations has been a standard in the industry for many years. In contrast to DiPrima and Handleman, Houbolt and Brooks restricted their analysis to small twist angles. In addition, it is assumed that the blade is initially stressed, and that stress is known. Because of the fact that extension was neglected early in their analysis, the corresponding equation was never formulated. For the purposes of this article, the extension equation has been reconstructed and appears as Eq. (3). Neglecting terms involving torque offset, the equations from Ref. 9 can be rewritten as

$$-T' - m\Omega^2 r = 0 \tag{3}$$

$$\begin{aligned} -(Tv')' &+ [Te_A(\cos\theta - \phi\sin\theta) - EB_2\theta'\phi'\cos\theta \\ &+ (EI_{z'}\cos^2\theta + EI_{y'}\sin^2\theta)v'' + (EI_{z'} \\ &- EI_{y'})w''\cos\theta\sin\theta]'' + m\ddot{v} - me\ddot{\phi}\sin\theta \\ &- 2me\Omega(\dot{v}'\cos\theta + \dot{w}'\sin\theta) - m\Omega^2[v + e(\cos\theta \\ &- \phi\sin\theta)] - [me\Omega^2 r(\cos\theta - \phi\sin\theta)]' = L_v \end{aligned} \tag{4}$$

$$\begin{aligned} -(Tw')' &+ [-Te_A(\cos\theta + \phi\sin\theta) \\ &- EB_2\theta'\phi'\sin\theta + (EI_{z'}\sin^2\theta + EI_{y'}\cos^2\theta)w'' \\ &+ (EI_{z'} - EI_{y'})v''\cos\theta\sin\theta]'' + m\ddot{w} \\ &- me\ddot{\phi}\cos\theta - [me\Omega^2 r(\sin\theta + \phi\cos\theta)]' = L_w \end{aligned} \tag{5}$$

$$\begin{aligned} -[Tk_A^2&(\theta + \phi)' + EB_1\theta'^2\phi' - EB_2\theta'(v''\cos\theta \\ &+ w''\sin\theta)]' - Te_A(w''\cos\theta - v''\sin\theta)' \\ &- (GJ\phi')' + mk_m^2\ddot{\phi} + m\Omega^2\phi(k_{m2}^2 - k_{m1}^2)\cos 2\theta \\ &+ me[\Omega^2 r(w'\cos\theta - v'\sin\theta) - (\ddot{v} - \Omega^2 v)\sin\theta \\ &+ \ddot{w}\cos\theta] = M_\phi - m\Omega^2(k_{m2}^2 - k_{m1}^2)\cos\theta\sin\theta \end{aligned} \tag{6}$$

Note that Eq. (3) can be solved for T and substituted into Eqs. (4–6) to eliminate T. Since Eq. (3) does not contain the extension variable, u in it, the original set of four equations in four unknowns, T, v, w, φ, can be reduced to three equations in three unknowns, v, w, φ. The nonlinear, Coriolis coupling terms that were neglected by DiPrima and Handleman also do not appear in Houbolt and Brooks equations.

Nonlinear Analyses

For uncoupled bending or torsion of a rotating beam, it has been common practice to use linear equations to model the beam dynamics. While such a set of equations is sufficient for most engineering applications involving flap (out-of-plane) or torsion deflections, linear equations neglect significant, nonlinear, gyroscopic terms in the lag (in-plane) equation. When flap and lag are coupled, a linear set of equations also neglects the Coriolis coupling between flap and lag. The importance of these effects was first discovered as a result of a rigid-blade analysis by Young.[21] These results were then amplified by Ormiston and Hodges.[22] The equations for the elastic flap-lag dynamics of a rotating beam were derived by Friedmann and Tong,[23] and by Hodges and Ormiston.[24]

To compare the linear equations derived by Houbolt and Brooks[9] with those derived by Hodges and Ormiston,[24] it will be necessary to neglect all of the terms in Houbolt and Brooks bending equations that contain the torsion variable and all offsets, as well as the entire torsion equation. Similarly, Hodges and Ormiston's equations must be modified to eliminate all of the terms that contain precone. The resulting bending equations may be written in the following form:

$$\begin{aligned} -(Tv')' &+ [(EI_{z'}\cos^2\theta + EI_{y'}\sin^2\theta)v'' \\ &+ (EI_{z'} - EI_{y'})w''\cos\theta\sin\theta]'' \\ &+ \underline{2m\Omega\dot{u}} + m\ddot{v} - m\Omega^2 v = L_v \end{aligned} \tag{7}$$

$$\begin{aligned} -(Tw')' &+ [(EI_{z'}\sin^2\theta + EI_{y'}\cos^2\theta)w'' \\ &+ (EI_{z'} - EI_{y'})v''\cos\theta\sin\theta]'' + m\ddot{w} = L_w \end{aligned} \tag{8}$$

The underlined term in Eq. (7) is a Coriolis term that is not present in Houbolt and Brooks equations. The time derivative of u is this equation can be eliminated by making the substitution

$$\dot{u} = -\underline{\int_0^r v'\dot{v}'\,dx} - \underline{\int_0^r w'\dot{w}'\,dx} \tag{9}$$

which is derived from the strain-displacement relation given in Ref. 24. The underlined terms in Eq. (9) are not present in the strain-displacement equation given in Ref. 9. When the extension equation given in Ref. 24 is solved for the tension

$$T = \Omega^2\int_r^R mx\,dx - \underline{2\Omega\int_r^R m\dot{v}\,dx} \tag{10}$$

where the underlined term again denotes a term that was neglected in Ref. 9. Equation (10) can be substituted into the first term of Eqs. (7) and (8) as before. After this substitution, the Coriolis term in Eq. (8) that corresponds to the one already derived from the underlined term in Eq. (7) can be seen.

One major problem that is encountered when making the extension to the coupled, nonlinear equations of motion for coupled bending and torsion of a rotating beam is that the equations quickly become very long and complicated. To make the equations more tractable, ordering schemes that could be used to eliminate small terms in the equations of motion were created. The basis of all commonly used ordering schemes is to define a parameter ε, which is the same order of magnitude as the bending slopes, and use it to estimate the magnitude of each term in the equation. Using the ordering scheme, a systematic set of rules can be adopted and used to determine which terms to retain, and which to neglect.

In the literature, many rotating beam equations of motion derivations use an ordering scheme that neglects terms of $\mathcal{O}(\varepsilon^2)$ relative to 1. One of the first, and best-known of these derivations was performed by Hodges and Dowell.[10] The derivation follows that of Ref. 9 very closely, but instead of simply neglecting nonlinear terms, it rigorously applies an ordering scheme to eliminate small terms. Other investigators have followed a similar approach in deriving rotor blade equations of motion. Included in this group are Hodges[25] and Sivaneri and Chopra.[26] In addition, Chopra and his coworkers at the University of Maryland have continued to capitalize on this approach in their development of a comprehensive analysis capability.

The similarities and differences between the linear equations in Ref. 9 and the nonlinear equations derived in Ref.

10 are easily seen by writing the two sets of equations using the same notation. To make this comparison easier, all of the terms which result from differences in the physical configurations assumed for each analysis have been removed. The following equations are identical to Eqs. 61a–d in Ref. 10, except that terms that include precone and warping have been deleted:

$$-T' - m\Omega^2 r + 2m\Omega\dot{v} = 0 \tag{11}$$

$$\begin{aligned} &-(Tv')' + \{-Te_A \cos(\theta + \phi) - EB_2\theta'\phi' \cos\theta \\ &+ [EI_{z'} \cos^2(\theta + \phi) + EI_{y'} \sin^2(\theta + \phi)]v'' \\ &+ (EI_{z'} - EI_{y'})w'' \cos(\theta + \phi)\sin(\theta + \phi)\}'' \\ &+ m\ddot{v} - me\ddot{\phi} \sin\theta + 2m\Omega\dot{u} - 2me\Omega(\dot{v}' \cos\theta \\ &+ \dot{w}' \sin\theta) - m\Omega^2[v + e\cos(\theta + \phi)] \\ &- \{me[\Omega^2 r \cos(\theta + \phi) + 2\Omega\dot{v} \cos\theta]\}' = L_v \end{aligned} \tag{12}$$

$$\begin{aligned} &-(Tw')' + \{-Te_A \sin(\theta + \phi) - EB_2\theta'\phi' \sin\theta \\ &+ [EI_{z'} \sin^2(\theta + \phi) + EI_{y'} \cos^2(\theta + \phi)]w'' \\ &+ (EI_{z'} - EI_{y'})v'' \cos(\theta + \phi)\sin(\theta + \phi)\}'' \\ &+ m\ddot{w} - me\ddot{\phi} \cos\theta - \{me[\Omega^2 r \sin(\theta + \phi) \\ &+ 2\Omega\dot{v} \sin\theta]\}' = L_w \end{aligned} \tag{13}$$

$$\begin{aligned} &-[Tk_A^2(\theta + \phi)' + EB_1\theta'^2\phi' - EB_2\theta'(v'' \cos\theta \\ &+ w'' \sin\theta)]' - Te_A(w'' \cos\theta - v'' \sin\theta) \\ &- (GJ\phi')' + (EI_{z'} - EI_{y'})[(w''^2 - v''^2)\cos\theta \\ &\times \sin\theta + v''w'' \cos 2\theta] + mk_m^2\ddot{\phi} + m\Omega^2\phi(k_{m2}^2 \\ &- k_{m1}^2)\cos 2\theta + me[\Omega^2 r(w' \cos\theta - v' \sin\theta) \\ &- (\ddot{v} - \Omega^2 v)\sin\theta + \ddot{w} \cos\theta] \\ &= M_\phi - m\Omega^2(k_{m2}^2 - k_{m1}^2)\cos\theta \sin\theta \end{aligned} \tag{14}$$

where

$$T = EA\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \tag{15}$$

Clearly, the linear equations, Eqs. (3–6), and the nonlinear equations, Eqs. (11–14), have many similarities, but there are significant differences between them. First, Eqs. (12–14) contain nonlinear, elastic coupling terms between bending and torsion that are not present in the linear equations. In Eqs. (12) and (13), these terms appear as sines and cosines of $(\theta + \varphi)$. The corresponding terms in Eq. (14) contain differences of the EI. Another difference between the two sets of equations is a result of the offset of the blade section c.m. from the elastic axis. In the nonlinear equations these terms produce Coriolis coupling between flap and lag, while there are no corresponding terms in the linear equations.

Another approach to the derivation of the nonlinear equations of motion was taken by Kaza and Kvaternik.[11,27] The major difference between the two analyses is the parameterization of elastic torsion. Kaza and Kvaternik use an orientation angle based on the sequence of rotations that describe the position of the deformed blade, while Hodges and Dowell use a quasicoordinate. In addition, Kaza and Kvaternik derive their equations using two different sequences of orientation angles and, therefore, arrive at two different sets of equations. Even though the different torsion definitions and sequences of rotations result in equations that do not appear to be the same, all are equivalent.

Naturally, there are many combinations of coordinate definitions and transformation sequences that could be used to derive the equations of motion for a rotating beam. Executed correctly, all are equivalent, but care must be taken when comparing them. Some of the problems encountered in such a comparison are discussed in detail in Ref. 28. In summary, the use of different sequences of orientation angles to derive equations of motion will result in equations that appear to be different. The reason that they are different is that the rotation parameters used in each set of orientation angles are physically different angles. In the two rotation sequences, flap-lag-torsion and lag-flap-torsion, the torsion angles are different angles by definition.

In the derivations discussed above, an ordering scheme was employed that neglected terms of $\mathcal{O}(\varepsilon^2)$ relative to 1. However, strict application would require that the torsion inertia term be dropped, since it is $\mathcal{O}(\varepsilon^3)$. Since that could result in a singular mass matrix, the term was retained. The importance of this small term raises the question whether any other small, but important terms have been neglected. Crespo da Silva,[13] using symbolic manipulation, derived a set of flap-lag-torsion equations for a rotating beam that retained terms up through $\mathcal{O}(\varepsilon^3)$. His methodology basically followed that of Hodges and Dowell. The use of symbolic manipulation to derive the equations is significant, since these equations are very long and complex. Reference 29 then demonstrated that both linear and nonlinear terms of $\mathcal{O}(\varepsilon^3)$ can significantly affect nonlinear equilibrium and linear stability. The most important of these terms are those associated with structural geometric nonlinearities in the torsion equation. Although not as potentially significant, the corresponding terms in the bending equations must be included to maintain symmetry in the stiffness matrix.

Exact Analyses

The use of an ordering scheme in the derivation of equations of motion requires great care on the part of the analyst. For the equations of motion of a rotating beam, the structural and inertial operators are self-adjoint. If the use of an ordering scheme results in a mass or stiffness matrix that is not symmetric, or a gyroscopic matrix that is not antisymmetric, the ordering scheme has destroyed one of the fundamental physical characteristics of the model.

Rotor blade analyses that employ ordering schemes to simplify the equations of motion must face two limitations in addition to those imposed by the assumptions on the physics of the problem. First, as pointed out by Stephens et al.,[30] it is impossible to apply an ordering scheme in a completely consistent manner. Many times, terms that should be neglected under the strict application of an ordering scheme, will have to be retained in order to satisfy known physical characteristics of a problem. For example, in the equations derived by Hodges and Dowell, the ordering scheme dictates that torsion inertia should be dropped from the torsion equation. Clearly, neglecting torsion inertia is not physically reasonable, as long as torsional motion is retained. In addition, Ref. 14 notes that ordering schemes are generally not desirable for general-purpose analyses in which the equations must be valid for all values of the configuration parameters. In all analyses that use an ordering scheme, the form of the equations is dependent on the magnitudes of the configuration parameters. Therefore, when the parameters exceed their prescribed bounds, the equations are no longer valid.

Hodges[14] developed a set of nonlinear equations for rotating, twisted beams that does not rely on an ordering scheme. Within the assumptions imposed on the physical model, the kinematics of the equations are exact. To keep the equations in a tractable form, the equations were derived in implicit form. While these equations retain the displacement formulation used in all of the analyses that employ ordering schemes, they appear to be simpler because of the implicit form. While this form of the equations allows the incorporation of exact kinematics without excessive complexity, it makes comparison with equations such as those derived by Hodges and Dowell nearly impossible (without extensive algebraic manipulations). In addition, the derivation of perturbation stiffness matrix for stability analyses cannot be performed analytically (without a prohibitive amount of work), and therefore must be formed numerically.

In all of the foregoing analyses, the equations of motion were expressed in second-order form, using displacement variables as the generalized coordinates. Using a mixed formulation, the equations of motion for beams undergoing large deformations were derived in first-order form by Simo[15] and applied beams undergoing large motions in space by Simo and Vu-Quoc.[31] These equations were derived such that the boundary conditions for the final equations are all natural. The advantage of this "weak" formulation is that the shape functions selected for the spatial discretization of the generalized coordinates are not forced to satisfy any geometric boundary conditions. Therefore, the shape functions can be selected to be very simple functions, for which integration is trivial. In addition, the equations of motion tend to be very sparse. The disadvantage of such a formulation is that many more differential equations must be solved to get accuracy equivalent to that obtained using a displacement formulation.

Hodges[16] has also derived a mixed variational formulation for the dynamics of moving beams. These equations extend the analysis used by Simo and Vu-Quoc[31] in their study of beams undergoing large motions in space. In Hodges' formulation, the rigid-body (frame) motions of the beam are prescribed kinematical variables, distinct from the beam elastic deformations. Simo and Vu-Quoc, on the other hand, have implicitly included the frame motion in the beam kinematical variables (intrinsic frame). The advantage of having separate frame kinematical variables (floating frame) is that the rigid body and elastic deformations can be identified and computed separately. However, the coupling between the rigid-body and elastic motion tends to be complex. The key to successfully implementing the intrinsic frame approach is to develop a procedure for computing the nonlinear internal beam forces, while preserving the frame motion.[32]

The equations derived in Ref. 16 cast the equations of motion for a beam into a canonical form, from which all of the foregoing analyses may be derived (as special cases)

$$\int_{t_1}^{t_2}\int_0^l \{(\delta\dot{q}^T - \delta q^T\tilde{\Omega} - \delta\psi^T\tilde{V})P + (\delta\dot{\psi}^T - \delta\psi^T\tilde{\Omega})H - [(\delta q')^T - \delta q^T\tilde{K} - \delta\psi^T(\tilde{e}_1 - \tilde{\gamma})]F - [(\delta\psi')^T - \delta\psi^T\tilde{K}]M + \delta q^T f + \delta\psi^T m\}\,\mathrm{d}x_1\,\mathrm{d}t = \int_0^l (\delta q^T\hat{P} + \delta\psi^T\hat{H})\big|_{t_1}^{t_2}\,\mathrm{d}x_1 - \int_{t_1}^{t_2}(\delta q^T\hat{F} + \delta\psi^T\hat{M})\big|_0^l\,\mathrm{d}t \tag{16}$$

where e_1 equals δ_{i1}, the $\delta(\ )$ are virtual displacements or velocities, the $\hat{(\ )}$ are discrete boundary values, and all quantities are expressed in the deformed beam coordinate system. Equation (16) is an exact intrinsic equation in weak form. All that is required to cast this equation into a solvable form is two constitutive relations and a kinematical relation. The first constitutive relation uses the kinetic energy to relate the generalized speeds (V and Ω) to the linear and angular momenta (P and H). The other relates the force and moment strains (γ and K) to the internal forces and moments (F and M) through the strain energy. Finally, the kinematical relation expresses the generalized displacements and rotations (q and ψ) in terms of a set of displacement and rotation variables. For engineering beam analyses, the kinematical relation and the constitutive law based on kinetic energy can be derived exactly. Therefore, all of the approximations in the equations of motion are contained in the other constitutive law, which is derived from the strain energy.

Composite Analyses

In the preceding sections, the development of the equations of motion for rotating beams focused on improving the kinematic representation of the beam motion. It was assumed, in general, that the beams being analyzed were constructed from a homogeneous, linearly elastic, isotropic material. An analysis of a built-up beam structure that is made up of composite, anisotropic materials requires consideration of the constitutive law for each material as well as the warping behavior of each cross section. The constitutive law for a linearly elastic, anisotropic material can be expressed in matrix form as

$$\{\sigma\} = [\bar{Q}]\{\varepsilon\} \tag{17}$$

where $[\bar{Q}]$ is a symmetric, 6×6 matrix with 21 independent constants that must be determined. For an isotropic material, there is considerable simplification since all but three, diagonal elements (with two independent constants) of this matrix become zero. In this section, the development of analysis methods that can model the behavior of built-up, anisotropic beams will be described.

To study the effect of stiffness coupling due to different ply lay-ups, Hong and Chopra[33] extended the equations derived by Hodges and Dowell[10] by replacing the original, isotropic stress-strain (constitutive) equations with the constitutive equations for a laminated structure. Only thin-walled, single-cell, rectangular box-beams were modeled, and cross-section warping was neglected. The constitutive equation for a single laminate was assumed to be that of a material with orthotropic properties in the direction of the fibers. In a later study[34] of flexbeams with I-beam cross sections, cross-sectional warping was implemented by predefining warping functions for the flanges and web. The function selected was the warping function for an elliptical bar in pure torsion.

While Hong and Chopra concentrated on the effect of ply lay-ups on the stiffness coupling, Bauchau[17] concentrated his efforts on modeling the warping behavior of thin-walled beams with closed cross sections. His approach involved improving Euler-Bernoulli beam theory by introducing transverse shear and a warping displacement, which was expanded in terms of a series of eigenwarpings. The other beam deformations were then corrected accordingly. The Saint-Venant solution was similarly improved by introducing the warping generalized coordinates into the definition of the axial stress flow. Bauchau assumed that there was no in-plane warping of cross sections, and that the beam was made up of materials that were either isotropic or transversely isotropic. In later work,[35,36] this assumption was relaxed to include generally orthotropic materials, and was applied to helicopter rotor blades using a finite element approach.

Stemple and Lee[37] took a somewhat different approach to modeling the out-of-plane warping of a beam undergoing small deformations. Instead of using a single warping generalized coordinate expanded into a series of modes, a number of warping nodes were placed in the cross-sectional plane to represent the warping deflection of points in the cross section. Like the beam analyses that have been discussed previously, this analysis assumes orthotropic material properties. However, through the use of the warping nodes, this analysis is capable of modeling restrained warping at the ends of the beam. Reference 38 later extended this analysis to model large deformations of the beam.

Kosmatka[39,40] has also used the finite element method to perform cross-sectional analysis of composite rotor blades. This analysis is applicable to beams with arbitrary cross-sectional geometry, including open cross sections. The cross-sectional analysis contains no restrictions on in-plane or out-of-plane warping, nor does it assume a uniaxial stress field. However, in developing the nonlinear equations of motion for the blade, the stress field is assumed to be uniaxial. Thus, in-plane warping is uncoupled from out-of-plane warping, and is then neglected. Like Bauchau's analysis, the materials may be generally orthotropic, but transverse shear is neglected and no independent warping generalized coordinate is introduced. In subsequent work, Kosmatka[41] has extended this analysis to include in-plane warping, thus accounting for Poisson contraction and anticlastic deformation due to bending.

All of the composite beam analyses that have been discussed to this point have made simplifying assumptions that limit their applicability to nonhomogeneous, anisotropic beams. In part, this is due to the fact that each of these methods has attempted to incorporate the cross-sectional properties into the beam equations of motion in explicit manner. Even with simplifying assumptions, this approach results in equations of motion that are extremely complex. As discussed above, Ref. 16 has shown that beam equations can be expressed in a canonical form, which implicitly contains the constitutive equations. The analyses that will be discussed in the remainder of this section assume this type of formulation either implicitly or explicitly.

In order to incorporate generalized warping, as well as anisotropy and nonhomogeneity into a beam cross-sectional analysis, Giavotto et al.[42] have developed a two-dimensional, finite element analysis that is capable of calculating generalized warping functions and cross-sectional properties for straight, untwisted beams with arbitrary cross-sectional geometries. Their formulation is based on the principle of virtual work. Starting with an expression for virtual work density for a beam of unit length, the cross section is first discretized into finite elements. Then, the cross-sectional displacements are separated into rigid and warping displacements, and a set of equilibrium equations is obtained. From these equations, the warping displacements at the ends of the beam segment, as well as in the center, can be calculated. The constitutive equation for the cross section is obtained by substituting the central equilibrium solution into the equilibrium equations, and taking the inverse of the resultant compliance matrix. This methodology is currently being extended to curved, twisted beams.[43]

Another approach to the modeling of composite beams has been developed by Atilgan and Hodges.[44] This approach is based on the weak Hamiltonian approach described by Hodges,[16] and it uses different strain measures than any of the previous analyses. By using the Cosserat strain measures, the constitutive law for the beam is written as a relationship between the (curvature-like quantities) force and moment strain, and the conjugate forces and moments. Engineering (Green's) strain defines a relationship between stress and strain. The advantage of using the Cosserat strain is that the conjugate forces and moments are the true forces and moments at the cross section. The cross-sectional analysis in this approach assumes both in-plane and out-of-plane warping, and transverse shear. Cross-sectional geometries are restricted to thin-walled, closed cross sections. The analysis permits large global rotations, small local rotations, and small strains.

An alternative formulation for the cross-sectional analysis of nonhomogeneous, anisotropic beams is presented in Refs. 45 and 46. This methodology is based on using the variational-asymptotical method to reduce the three-dimensional strain field to an approximate one-dimensional field. The resulting asymptotically correct stiffnesses can then be used in the nonlinear equations of motion for a composite beam. An application of this method to static and dynamic beam response is shown in Ref. 47.

Concluding Remarks

Engineering beam theories that are applicable to helicopter rotors have been traced through several significant phases. The first began with the recognition that bending flexibility was an important factor in the analysis of rotating beams. Torsion was added somewhat later, to complete the development of linear equations for the coupled bending and torsion of rotating beams. The second phase of development added nonlinear terms to the equations of motion, and introduced ordering schemes order to reduce the complexity of the equations. In the third phase, ordering schemes have given way to the exact equations of motion, often written in compact notation and employing mixed formulations. The latest phase in the development of rotating beam theories has encompassed the addition of the effects of anisotropic material properties that are introduced by the use of composite materials in rotor blades and bearingless rotor flexbeams.

For the engineering analysis of helicopter rotor blades, it is essential that the analyst understand both the complexity of the structures that he is trying to analyze, and the capabilities of the analysis methods that he has at his disposal. He will obtain maximum efficiency for his effort by matching the solution method as closely as possible to the problem at hand. It makes no more sense to use a composite beam analysis with its finite element cross-sectional analysis to analyze a solid, wood, articulated rotor blade, than it does to study the response of a composite, bearingless rotor with a simple flapping blade analysis. This survey of the state-of-the-art in rotating beam analysis provides practicing engineers with an overview of the available methodologies from which they can choose.

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